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Cutoff dependence of the Casimir effect

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Abstract. The problem of calculating the Casimir force on two conducting planes by means of the stress tensor is examined. The evaluation of this quantity is carried out using an explicit regularization procedure which has its origin in the underlying (2+1) dimensional Poincaré invariance of the system. The force between the planes is found to depend on the ratio of two independent cutoff parameters, thereby rendering any prediction for the Casimir effect an explicit function of the particular calculational scheme employed. Similar results are shown to obtain in the case of the conducting sphere.

In 1948 Casimir [1] first predicted that two infinite parallel plates in vacuum would attract each other. This remarkable result has its origin in the zero point energy of the electromagnetic field. While the latter is highly divergent, the change associated with this quantity for specific plate configurations has been found in numerous calculations to be finite as well as cutoff dependent and thus in principle observable. Early work to detect this small effect [2] was characterized by relatively large experimental uncertainties which left the issue in some doubt. More recent efforts [3] have provided quite remarkable data, but are based on a different geometry from that of Casimir. Since a rigorous theoretical calculation has never been carried out for the latter configuration, there remains room for skepticism as to whether the Casimir effect is as well established as is frequently asserted.

The most elementary calculation of the Casimir effect between two parallel conducting planes located at $z = 0$ and $z = a$ employs a mode summation in the framework of a regularization which depends only on the frequency $\omega_k = [\mathbf{k}^2 + (\frac{n\pi}{a})^2]^{\frac{1}{2}}$ where $n = 0, 1, 2, ...$ Upon combining the result obtained with the corresponding result for the interval $a \leq z \leq L$ where $L \gg a$ is the z-coordinate of a third conducting plane, a finite cutoff independent result[1] $F/A = \pi^2/240a^4$ is obtained for the Casimir pressure on the plate at $z = a$.

A considerably more elegant approach to this problem is that of Brown and Maclay [4] who employ an image method to calculate $\langle 0|T^{\mu\nu}(x)|0\rangle$. Thus they showed that the photon propagator in the presence of conducting planes at $z = 0$ and $z = a$ could be expressed in terms of an infinite sum over the usual (i.e., $-\infty < z < \infty$) photon propagator with the z-coordinate of each term in the sum displaced by an even multiple of a. Since the stress tensor for the electromagnetic case is given by

$$T^{\mu\nu}(x) = F^{\mu\alpha}F^{\nu}_{\ \alpha} - \frac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta} \tag{1}$$

where

$$F^{\mu\nu}(x) = \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x),$$

it follows that upon taking appropriate derivatives with respect to the propagator arguments x and x' and invoking the limit $x \to x'$ a formal expression can be obtained for the vacuum expectation value of the stress tensor. On the basis of covariance arguments together with the divergence and trace free property of $T^{\mu\nu}(x)$ it was then found in [4] that

$$\langle 0|T^{\mu\nu}(x)|0\rangle = \left(\frac{1}{4}g^{\mu\nu} - \hat{z}^\mu\hat{z}^\nu\right)\left(\frac{1}{2\pi^2a^4}\right)\sum_{n=1}^{\infty} n^{-4} \tag{2}$$

where $\hat{z}^\mu$ is the unit vector (0,0,1,0) in the z-direction normal to the conducting planes.

However, there is some reason to question whether this approach has adequately dealt with the divergences which invariably occur in Casimir calculations. One notes in particular that the result (2) is obtained only after an obviously singular $n = 0$ term has been dropped from the sum which occurs in that equation. While one can argue as in [4] that such an a-independent term can be freely omitted since it is merely the usual subtraction of the large a result, it is well to note that the *entire* sum over n is required for a demonstration that the propagator satisfies correct boundary conditions at $z = 0, a$. Moreover, as is shown in this work, an appropriately regularized form of (2) does not necessarily allow a separation into cutoff dependent terms and a-dependent terms, in contrast with the result found in [4]. Of still greater import is the fact that more general regularizations than those usually considered in this calculation lead to an explicit cutoff dependence of

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[1] The units employed here are such that $\hbar = c = 1$ with the signature of the metric being $(1, 1, 1, -1)$.

the Casimir stress, a circumstance which would seem to deny its physical significance.

To establish the above claims one reverts from the image approach to one based on expansion of the Green's function in terms of orthogonal functions[2]. To this end one notes that the free field propagator in the radiation gauge can be written as

$$G^{ij}(\mathbf{x}-\mathbf{x}', z, z', t-t') = \sum_{n\lambda} \int \frac{d\mathbf{k}d\omega}{(2\pi)^3} e^{-i\omega(t-t')} \times \frac{A^i_{n\lambda}(\mathbf{k}, z) A^{j*}_{n\lambda}(\mathbf{k}, z')}{k^2 - \omega^2 + (n\pi/a)^2 - i\epsilon} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} \quad (3)$$

where $\lambda = 1, 2$ refers to the polarization, and spatial coordinates orthogonal to the z-direction are denoted by a boldface notation. The eigenfunctions $A^i_{n\lambda}(\mathbf{k}, z)$ satisfy the equation

$$\left[\frac{\partial^2}{\partial z^2} + (n\pi/a)^2\right] A^i_{n\lambda}(\mathbf{k}, z) = 0$$

and are given explicitly by

$$A^i_{n1}(\mathbf{k}, z) = \frac{\overline{k}_i}{|\mathbf{k}|}\left(\frac{2}{a}\right)^{\frac{1}{2}} \sin(n\pi z/a) \quad (4)$$

and

$$A^i_{n2}(\mathbf{k}, z) = \frac{1}{|\mathbf{k}|\omega_k}\left(\hat{\mathbf{z}}^i\omega_k^2 + \hat{\mathbf{z}}\cdot\nabla\nabla^i\right)\left(\frac{2}{a}\right)^{\frac{1}{2}} \cos(n\pi z/a) \quad (5)$$

where $\omega_k^2 = \mathbf{k}^2 + (n\pi/a)^2$ and $\overline{k}_i \equiv \epsilon^{ij}k_j$ with ϵ^{ij} being the usual alternating symbol. (In the $n = 0$ case the rhs of (5) must be multiplied by a factor of $2^{-\frac{1}{2}}$.) It is important to note that each eigenfunction $A^i_{n\lambda}(\mathbf{k}, z)$ satisfies the boundary conditions $\hat{\mathbf{z}} \times \mathbf{E} = \hat{\mathbf{z}} \cdot \mathbf{B} = 0$ at $z = 0, a$. This means that it is possible to introduce a regularization such that contributions from large values of $|\mathbf{k}|$ and/or n are reduced without destroying the validity of the boundary conditions. This stands in marked contrast with the image method which has no mechanism for the consistent suppression of the contributions of higher order reflections.

In order to determine the regularization appropriate to this calculation one should ideally make reference to the underlying symmetry. Since the latter consists of the reflection $z \to a - z$ and the (2+1) dimensional Poincaré group, it is natural to seek to classify regularization schemes according to representations of the latter. The usual cutoff method for this problem invokes a parameter which damps out the large ω_k contributions, an approach which makes no reference to the underlying Lorentz invariance. A far more appropriate technique is to generalize this to a cutoff based on a vector σ^μ in (2+1) dimensions as well as a scalar cutoff Σ which can be used to suppress large values of the (2+1) dimensional invariant $E^2 - \mathbf{P}^2$ where E and $\mathbf{P}$ are respectively the energy and momentum operators associated with this (2+1) dimensional subspace. Clearly, the credibility of the Casimir effect requires that the result be independent of the relative importance of these two competing cutoffs.

The calculation proceeds by noting that since the limit $x \to x'$ is to be taken symmetrically at some point, it is appropriate to use only the imaginary part of the propagator. An appropriately regularized version of this function can be inferred from (3) to be[3]

$$\Im\, G^{ij}_{\sigma,\Sigma}(x, x') = \pi \sum_{n\lambda} \int \frac{d^3k}{(2\pi)^3} \delta(k^2 + (n\pi/a)^2) \times A^i_{n\lambda}(\mathbf{k}, z) A^{j*}_{n\lambda}(\mathbf{k}, z') e^{ik^\mu(x-x')_\mu} e^{\sigma_\mu k^\mu \epsilon(k^0)} e^{\Sigma(-k^2)^{\frac{1}{2}}} \quad (6)$$

where $\epsilon(k^0)$ is the alternating function and a summation convention has been introduced in the Lorentz invariant subspace. Note that since both σ^μ and k^μ are three vectors in that space, they satisfy the orthogonality conditions $\hat{z}^\mu\sigma_\mu = \hat{z}^\mu k_\mu = 0$[4]. In addition it is clearly necessary to impose $\overline{\sigma}^2 \equiv -\sigma^\mu\sigma_\mu > 0$ and $\sigma^0 > 0$ in order that this propagator exist. It will subsequently be found that its existence also requires $\Sigma < \overline{\sigma}$.

To proceed one uses the regularization (6) and the form of the stress tensor (1). When used in conjunction with the eigenfunctions (4) and (5) the vacuum expectation value of the regularized stress tensor can be determined. With some effort this is found by straightforward calculation to yield the coordinate independent result

$$\langle 0 \mid T^{\mu\nu} |0\rangle = \frac{2\pi}{a} {\sum_{n=0}^{\infty}}' \int \frac{d^3k}{(2\pi)^3} \delta(k^2 + (n\pi/a)^2) e^{\sigma_\mu k^\mu \epsilon(k^0)} \times e^{\Sigma n\pi/a}\left[k^\mu k^\nu + \hat{z}^\mu\hat{z}^\nu(n\pi/a)^2\right]$$

which is manifestly both symmetric and traceless. The prime on the summation denotes the fact that the $n = 0$ term must be multiplied by $\frac{1}{2}$ as a consequence of the normalization of $A^i_{02}(\mathbf{k}, \mathbf{z})$. It can be more usefully written as

$$\langle 0 \mid T^{\mu\nu} |0\rangle = \frac{1}{a} {\sum_{n=0}^{\infty}}' e^{\Sigma n\pi/a} \times \left(\frac{\partial}{\partial\sigma_\mu}\frac{\partial}{\partial\sigma_\nu} - \hat{z}^\mu\hat{z}^\nu\frac{\partial^2}{\partial\sigma^\alpha\partial\sigma_\alpha}\right) \Delta^{n\pi/a}(-i\sigma) \quad (7)$$

where $\Delta^{n\pi/a}(x)$ is the (2+1) dimensional function

$$\Delta^{n\pi/a}(x) = 2\pi \int \frac{d^3k}{(2\pi)^3} e^{ikx\epsilon(k^0)} \delta(k^2 + (n\pi/a)^2)$$

for a particle of mass $n\pi/a$. Since this is an $O(2,1)$ scalar, $\Delta^{n\pi/a}(-i\sigma)$ is a function of only the invariant $\overline{\sigma}$ which

[2] This approach has been used for the case of the sphere in C. R. Hagen, Phys. Rev. D **61**, (2000) 065005.

[3] Equivalence to the usual regularization would require that $\sigma_i = \Sigma = 0$.

[4] To do otherwise would also introduce complications associated with the fact that $\hat{z}^\mu P_\mu$ does not commute with E and $\mathbf{P}$.

has the explicit form

$$\Delta^{n\pi/a}(-i\sigma) = \frac{1}{2\pi\overline{\sigma}} e^{-\overline{\sigma} n\pi/a}.$$

The insertion of this result into (7) clearly implies that the sum over n exists only for the case that $\Sigma < \overline{\sigma}$ as previously stated. Upon performing the summation over n it follows that

$$\langle 0|T^{\mu\nu}|0\rangle = (\frac{\partial^2}{\partial\sigma_\mu\partial\sigma_\nu} - \hat{z}^\mu\hat{z}^\nu \frac{\partial^2}{\partial\sigma^\alpha\partial\sigma_\alpha})F(\overline{\sigma}, \Sigma)$$

where

$$F(\overline{\sigma}, \Sigma) = \frac{1}{4\pi a\overline{\sigma}} \coth \frac{(\overline{\sigma} - \Sigma)\pi}{2a}.$$

One now carries out the expansion of this expression discarding terms which give no contribution in the limit of vanishing cutoff, thereby obtaining

$$F(\overline{\sigma}, \Sigma) \to \left[\frac{1}{2\pi^2}\frac{1}{\overline{\sigma}}\frac{1}{\overline{\sigma} - \Sigma} - \frac{\Sigma}{24a^2\overline{\sigma}} - \frac{(\overline{\sigma} - \Sigma)^3\pi^2}{1440\overline{\sigma}a^4}\right].$$

Upon performing the derivatives and rearranging terms there finally results[5,6]

$$\begin{aligned}
&\langle 0|T^{\mu\nu}|0\rangle \\
&= \left[g^{\mu\nu} + 3\frac{\sigma^\mu\sigma^\nu}{\overline{\sigma}^2} - \hat{z}^\mu\hat{z}^\nu\right]\left\{-\frac{\Sigma}{24a^2\overline{\sigma}^3}\right. \\
&\left. + \frac{(2\overline{\sigma} - \Sigma)(\overline{\sigma} - \Sigma) + \frac{2}{3}\overline{\sigma}^2}{2\pi^2\overline{\sigma}^3(\overline{\sigma} - \Sigma)^3} + \frac{\pi^2}{1440a^4}\frac{\Sigma}{\overline{\sigma}}\left(\frac{\Sigma^2}{\overline{\sigma}^2} - 1\right)\right\} \\
&+ \left(\frac{1}{4}g^{\mu\nu} - \hat{z}^\mu\hat{z}^\nu\right)\left[\left(1 - \frac{\Sigma}{\overline{\sigma}}\right)\frac{\pi^2}{180a^4} - \frac{4}{3\pi^2}\frac{1}{\overline{\sigma}}\frac{1}{(\overline{\sigma} - \Sigma)^3}\right].
\end{aligned}$$

If (following [4]) one subtracts the $a \to \infty$ result, this reduces to the more tractable form

$$\begin{aligned}
\langle 0|\overline{T}^{\mu\nu}|0\rangle &= \left[g^{\mu\nu} + 3\frac{\sigma^\mu\sigma^\nu}{\overline{\sigma}^2} - \hat{z}^\mu\hat{z}^\nu\right] \\
&\quad \times\Sigma\left\{-\frac{1}{24a^2\overline{\sigma}^3} + \frac{\pi^2}{1440a^4\overline{\sigma}}\left(\frac{\Sigma^2}{\overline{\sigma}^2} - 1\right)\right\} \\
&\quad + \left(\frac{1}{4}g^{\mu\nu} - \hat{z}^\mu\hat{z}^\nu\right)\left(1 - \frac{\Sigma}{\overline{\sigma}}\right)\frac{\pi^2}{180a^4}
\end{aligned}$$

where an overbar notation has been used to denote this subtraction. It is noteworthy that even this removal of the large a result does not lead to regularization independent results, a fact which has been remarked upon earlier.

Of particular interest to Casimir calculations are the stress components $\langle 0|\overline{T}^{33}|0\rangle$ and the energy density per unit area $\mathcal{E} \equiv a\langle 0|\overline{T}^{00}|0\rangle$ which are given by

$$\langle 0|\overline{T}^{33}|0\rangle = -\frac{\pi^2}{240a^4}\left(1 - \frac{\Sigma}{\overline{\sigma}}\right) \tag{8}$$

and

$$\begin{aligned}
\mathcal{E} = -\frac{\pi^2}{720a^3}&\left\{1 - \frac{\Sigma}{\overline{\sigma}} - \frac{3\sigma_0^2 - \overline{\sigma}^2}{2\overline{\sigma}^3}\Sigma\right. \\
&\left.\times\left[\frac{\Sigma^2}{\overline{\sigma}^2} - 1 - \frac{30a^2}{\pi^2\overline{\sigma}^2}\right]\right\}
\end{aligned} \tag{9}$$

respectively. It is striking that each of these terms retains a significant dependence on the cutoff details. In addition the usual relation assumed (as in [4]) to hold between $\mathcal{E}$ and the stress components, namely

$$\langle 0|\overline{T}^{33}|0\rangle = -\frac{\partial}{\partial a}\mathcal{E}, \tag{10}$$

is manifestly contradicted by (8) and (9) in agreement with results found earlier in the context of the Casimir energy of a sphere[2]. It is significant that the relation (10) asserts a relationship between the vacuum stress $\langle 0|\overline{T}^{33}|0\rangle$ which transforms under $O(2,1)$ as a scalar while the right hand side transforms as the $\mu = \nu = 0$ component of a symmetric tensor under this group. Finally, note should be made of the fact that (9) predicts an additional Casimir force proportional to the divergent indeterminate form $\Sigma/a^2\overline{\sigma}^3$.

To reinforce the conclusions reached here in the case of parallel plates it is useful to consider also the case of the conducting sphere, the only other geometry in three dimensions which has proved amenable to exact calculation[7]. This case was first solved by Boyer [5] and subsequently verified by a number of authors [6–9]. Following reasoning similar to that of the parallel plate case note is made of the fact that the unbroken symmetry in this case consists of time translation and rotational invariance. Thus the natural cutoff parameters in this problem should

[5] It is of interest to note that from the metric $g^{\mu\nu}$ and the vectors σ^μ and $\hat{z}^\mu$ one can form three second rank tensors which subsequently reduce to two when the tracelessness condition is applied. The fact that there is no term of the form $(\sigma^\mu\hat{z}^\nu + \sigma^\nu\hat{z}^\mu)(a + b\hat{z}^\mu\sigma_\mu)$ is a consequence of the orthogonality condition $\hat{z}^\mu\sigma_\mu = 0$ and the invariance under $\hat{z}^\mu \to -\hat{z}^\mu$.

[6] It should be noted here that the mode summation approach for parallel plate geometry has been carried out in the case $\sigma^\mu = -i\Lambda^{-1}\delta_0^\mu$, $\Sigma = 0$ by B. DeWitt, Phys. Reports **19**C, (1975) 295. In view of the fact that such a cutoff singles out the time axis it is not surprising that the result obtained there for the vacuum stress is the explicitly *noncovariant* form

$$\frac{\Lambda^4}{\pi^2}(g^{\mu\nu} + 4\delta_0^\mu\delta_0^\nu) + \frac{\pi^2}{720a^4}(g^{\mu\nu} - 4\hat{z}^\mu\hat{z}^\nu).$$

[7] This leaves the case of one dimension as the only remaining example of a Casimir effect calculation which is regularization independent. In that application it could hardly be otherwise since the leading singularity is proportional to the square of an inverse cutoff parameter which is necessarily removed by subtraction of the $a \to \infty$ vacuum. In addition the next to leading term gives no effect since at most it could contribute a divergent a independent term to $\mathcal{E}$, thereby leaving a finite cutoff independent remainder to contribute to the force. In (3+1) dimensions, however, there are simply too many too many divergences, too many cutoff parameters, and too few physically reasonable subtractions to obtain a finite cutoff independent result.

refer to the energy and angular momentum. The former is the standard one and is well known to give cutoff independent results. It will be the goal here to examine the situation which occurs when a combination of these two is considered.

This is most economically achieved through[2] a useful separation of the Casimir energy into a finite part and one which requires regularization. Thus one writes for a sphere of radius a

$$E_c = E_{fin} + E_\sigma$$

where E_c, E_{fin}, and E_σ are respectively the total, the regularization independent part, and the formally divergent parts of the Casimir energy. The quantity E_σ is given by

$$E_\sigma = \frac{1}{4\pi a}\sum_{l=1}^{\infty}\Re e^{-i\phi}\int_0^\infty dy\exp(-i\nu\sigma y e^{-i\phi})y\frac{d}{dy} \times(1+y^2e^{-2i\phi})^{-3}$$

where $\nu = l + \frac{1}{2}$, $0 < \phi < \frac{\pi}{2}$, and σ is a dimensionless cutoff used to suppress the high frequency modes. Upon choosing a secondary cutoff of the form $e^{-\Sigma\nu}$ it is readily found that ΔE_σ (the *change* induced in E_σ in the limit of small cutoff) is given by

$$\Delta E_\sigma = -\frac{3\Sigma}{2\pi a\sigma^2}\int_0^\infty dy\frac{y^2}{(1+y^2)^4}\frac{1}{y^2+(\Sigma^2/\sigma^2)}.$$

This is evaluated to yield

$$\Delta E_\sigma = -\frac{3}{64a}\frac{\Sigma}{(\Sigma+\sigma)^4}[\Sigma^2+4\sigma\Sigma+5\sigma^2],$$

a result which displays yet again the cutoff dependence of the Casimir effect for a more general choice of regularization. It may be noted that aside from confirming the vanishing of ΔE_σ for $\Sigma = 0$, this result shows that ΔE_σ goes as Σ^{-1} for $\sigma \to 0$ with intermediate values being obtained for finite Σ/σ.

In this work it has been shown that the Casimir effect is, prevailing opinion notwithstanding, highly dependent on the particular form of regularization employed for the extraction of the force. As remarked earlier the recent experiments which have seemed to many to provide the long awaited precision verification of this highly subtle effect are not based upon rigorous mathematical calculation. While the parallel plate Casimir experiment is fraught with difficulties beyond the ken of this author, it would seem that the successful completion of such experiments would be invaluable for purposes of setting to rest some of the issues which have been raised in this work.

Finally, it would be remiss not to mention in some way the very extensive work on the calculation of Casimir forces using the technique of zeta function regularization [10]. Historically, the successes of the Casimir approach in dealing with the parallel plate geometry and the sphere were obtained using conventional field theoretical subtraction procedures. Specifically, it was noted that only changes *relative* to the vacuum could be considered observable and it was therefore totally consistent to perform subtractions relative to the $a \to \infty$ vacuum. However, this step did not succeed in allowing one to obtain finite and observable results in more general applications. Eventually it was realized, however, that the application of zeta function regularization to such problems could yield finite results for some fairly general cases while at the same time agreeing with those obtained in the few instances in which more conventional subtractions could be successfully applied. This work makes no claim to having established any inconsistencies in the derivation of finite results for the Casimir effect when those efforts are based on the twin axioms of vacuum energy *and* zeta function regularization. Rather, the calculations presented here establish that the Casimir effect is generally cutoff dependent and hence incapable of being reliably determined whenever such calculations are performed using conventional (i.e., physically plausible) subtraction procedures.

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